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Gauge structure of vacuum string field theory

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Abstract: We study the gauge structure of vacuum string field theory expanded around the D-brane solution, namely, the gauge transformation and the transversality condition of the massless vector fluctuation mode. We find that the gauge transformation on massless vector field is induced as an anomaly; an infinity multiplied by an infinitesimal factor. The infinity comes from the singularity at the edge of the eigenvalue distribution of the Neumann matrix, while the infinitesimal factor from the violation of the equation of motion of the fluctuation modes due to the regularization for the infinity. However, the transversality condition cannot be obtained even if we take into account the anomaly contribution.

Keywords: Bosonic Strings, Tachyon Condensation, String Field Theory, Gauge Symmetry.

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Contents

1. Introduction

It has been conjectured that vacuum string field theory (VSFT) [1]–[4] represents a string field theory expanded around the non-perturbative vacuum on which open string tachyon has condensed. The VSFT action is obtained from the action of the cubic string field theory (CSFT) by replacing the BRST operator Q_{B} with a purely ghost operator $\mathcal{Q}$. In order to check this conjecture, a classical solution of VSFT was constructed, which is expected to correspond to a D-brane. There have been many attempts to examine whether this solution has desired properties as a D-brane. The criteria whether it corresponds to a D-brane includes the tension of the solution and string spectrum around it. In the oscillator formalism, the classical solution is constructed as a squeezed state [5, 2, 6] and the mass spectrum around this solution is reproduced correctly [6, 7]. However, the correct D-brane tension has not been obtained in the oscillator formalism.[1]

In addition to the above criteria, the theory around a D-brane must have the gauge structure which is a significant property of string theory. To see this, let us recall the case of the free SFT which has the Kato-Ogawa BRST operator Q_{B}. Taking $b_0 b_{-1}|0\rangle\lambda$ as the state of the gauge transformation, we have[2]

$$Q_{\mathrm{B}}(b_0 b_{-1}|0\rangle\lambda) = b_{-1}|0\rangle\Box\lambda + b_0\alpha_{-1}^{\mu}|0\rangle\partial_\mu\lambda\,. \tag{1.1}$$

The second term is proportional to the massless vector state and generates the gauge

[1] See [8] for a derivation of the correct D-brane tension in the boundary conformal field theory approach.

[2] The state $|0\rangle$ is the Fock vacuum which satisfies $(\alpha_n^\mu, b_n, c_n)|0\rangle = 0$ for $n \geq 1$, and has no attributes of the zero-modes.

transformation $\delta A_\mu = \partial_\mu \lambda$ on the massless vector field A_μ. On the other hand, let us consider the equation of motion for the massless vector state with polarization ζ_μ:

$$0 = Q_{\mathrm{B}}(b_0 \alpha^\mu_{-1}|0\rangle \zeta_\mu) = \alpha^\mu_{-1}|0\rangle \Box \zeta_\mu + b_0 c_{-1}|0\rangle \partial^\mu \zeta_\mu \,. \tag{1.2}$$

We obtain the transversality condition $\partial_\mu \zeta^\mu = 0$ from the last term, while the first one provides the on-shell condition for ζ_μ. If the classical solution of VSFT represents a D-brane, VSFT expanded around it must reproduce the gauge transformation and the transversality condition of the massless vector state (and also of the massive higher level states).

The purpose of this paper is to investigate whether these kinds of gauge structure appear in VSFT using its oscillator formulation. This is in fact a non-trivial problem since a naive analysis around the classical solution using the oscillator formulation leads to the conclusion that neither the gauge transformation nor the transversality condition can arise from VSFT. We have already seen the difficulty in reproducing the transversality condition of massless vector state in [6], and the situation is almost the same also for the gauge transformation. In this paper, we reexamine the gauge structure concerning the massless vector state in detail and find that important terms were overlooked in the naive analysis; an infinite factor multiplied by an infinitesimal one. The infinite factor originates from the singularity at the edge of the eigenvalue distribution of the Neumann matrices defining the $*$-product of string fields [9, 10]. The same kind of singularity has made various observables in VSFT non-trivial ones. On the other hand, the infinitesimal factor is due to the failure of the equation of motion of fluctuation modes in the regularization introduced for controlling the infinity [11].

In our analysis of the gauge structure, the anomaly like (namely, $\infty \times 0$) contribution mentioned above indeed leads to the desired gauge transformation of the massless vector field in VSFT. The equation of motion of the massless vector state Φ_{v}, $\mathcal{Q}_{\mathrm{B}}\Phi_{\mathrm{v}} = 0$ with $\mathcal{Q}_{\mathrm{B}}$ being the BRST operator of VSFT around the classical solution, also receives the anomaly correction. However, contrary to the case of (1.2) which consists of two independent terms, the on-shell condition and the transversality condition, $\mathcal{Q}_{\mathrm{B}}\Phi_{\mathrm{v}} = 0$ is satisfied only by the on-shell condition. The role of the anomaly term is merely to shift the gauge of the on-shell condition and it cannot generate the transversality condition. Therefore, our finding in this paper is that the gauge transformation on the massless vector field is indeed generated as an anomaly, but the absence of the transversality condition still remains a mystery to be resolved. Our analysis in this paper is restricted to the Fock space, namely, for all the ket equations we are implicitly considering their inner product with any Fock space bra states. The extension to larger space including the sliver states is also our future problem.

Before closing this section we shall give two comments concerning the gauge structure in VSFT. First, as seen from (1.1) and (1.2), the ghost sector of states as well as the BRST operator plays a vital role in the gauge structure of the ordinary string field theory. On the other hand, in VSFT with purely ghost BRST operator, the ghost and matter sectors are factorized both in the classical solution and the fluctuation modes. Due to this property, the ghost sector has played no significant roles in the physics around the D-brane solution. The gauge structure of VSFT is interesting since this is the first place where the ghost sector makes an essential contribution.

Our second comment is on another role of gauge transformation in VSFT. The gauge transformation in VSFT has been used in the construction of massive and massless states around the D-brane solution. There, an infinite number of spurious states particular to VSFT have been gauged away by a special kind of gauge transformation [12, 7]. This gauge transformation is of completely different type from that we shall study in this paper.

The organization of the rest of this paper is as follows. In the next section, we summarize the VSFT action, its classical solutions and fluctuation modes. In section 3, we recapitulate the treatment of the singular behavior of the infinite dimensional matrices in the oscillator formulation of VSFT. This singular behavior plays a crucial role in the following analysis. Section 4 is the main part of this paper, and we investigate the gauge structure of the VSFT expanded around the D-brane solution. The final section (section 5) is devoted to a summary and discussions. In appendix A, B and C, we summarize several technical details used in the text.

2. VSFT in the oscillator formalism

In this section, we review the VSFT action, its classical solution and tachyon wave function as a fluctuation around the classical solution. The VSFT action proposed in [1, 2, 4] is

$$\begin{aligned} S &= -K\left(\frac{1}{2}\Psi\cdot\mathcal{Q}\Psi+\frac{1}{3}\Psi\cdot(\Psi*\Psi)\right) \\ &= -K\left(\frac{1}{2}\int_{b_0,x}\langle\Psi|\mathcal{Q}|\Psi\rangle+\frac{1}{3}\int_{b_0,x}^{(1)}\int_{b_0,x}^{(2)}\int_{b_0,x}^{(3)}{}_1\langle\Psi|_2\langle\Psi|_3\langle\Psi|V\rangle_{123}\right), \end{aligned} \tag{2.1}$$

where K is a constant, and the integration $\int_{b_0,x}^{(r)}\equiv\int db^{(r)}\int d^{26}x_r$ is over the zero modes of the r-th string. The BRST operator $\mathcal{Q}$ of VSFT consists purely of ghost oscillators,

$$\mathcal{Q}=c_0+\sum_{n=1}^{\infty}f_n\bigl(c_n+(-1)^n c_n^\dagger\bigr)\,, \tag{2.2}$$

with f_n being a constant. The VSFT action (2.1) has an invariance under the gauge transformation:

$$\delta_\Lambda\Psi=\mathcal{Q}\Lambda+\Psi*\Lambda-\Lambda*\Psi\,. \tag{2.3}$$

The $*$-product in (2.1) and (2.3) is the same as in the CSFT action and defined through the three string vertex $|V\rangle$ which is given in the momentum representation by

$$\begin{aligned} |V\rangle_{123} &= \exp\left\{\sum_{r,s=1}^{3}\left(-\sum_{n,m\geq 0}\frac{1}{2}a_n^{(r)\dagger}V_{nm}^{rs}a_m^{(s)\dagger}+\sum_{n\geq 1,m\geq 0}c_n^{(r)\dagger}\widetilde{V}_{nm}^{rs}b_m^{(s)\dagger}\right)\right\}|0\rangle_{123}\times \\ &\quad\times(2\pi)^{26}\delta^{26}(p_1+p_2+p_3)\,, \end{aligned} \tag{2.4}$$

where $a_0^{(r)}=a_0^{(r)\dagger}=\sqrt{2}p_r$ is the center-of-mass momentum. The coefficients V_{nm}^{rs} of the oscillators are infinite dimensional matrices and are referred to as Neumann coefficients. It is convenient to introduce the following new matrices and vectors:

$$[M_0]_{mn} = [CV^{r,r}]_{mn}\,, \qquad [M_\pm]_{mn} = [CV^{r,r\pm1}]_{mn}\,, \tag{2.5}$$

$$[\boldsymbol{v}_0]_n = [V^{r,r}]_{n,0}\,, \qquad [\boldsymbol{v}_\pm]_n = [V^{r,r\pm1}]_{n,0}\,, \tag{2.6}$$

$$V_{00} = [V^{r,r}]_{00} = \frac{1}{2}\ln\left(\frac{3^3}{2^4}\right), \tag{2.7}$$

where m and n run from 1 to infinity, and C is the twist matrix:

$$C_{mn} = \delta_{mn}(-1)^n\,. \tag{2.8}$$

The Neumann matrices and vectors in the ghost sector are denoted by adding a tilde to the corresponding one in the matter sector; $\widetilde{M}_{0,\pm}$ and $\widetilde{\boldsymbol{v}}_{0,\pm}$. We often use the following combination of the Neumann coefficients:

$$M_1 \equiv M_+ - M_-\,, \tag{2.9}$$

$$\boldsymbol{v}_1 \equiv \boldsymbol{v}_+ - \boldsymbol{v}_-\,. \tag{2.10}$$

Note that M_1 and $\boldsymbol{v}_1$ are twist-odd, while M_0 and $\boldsymbol{v}_0$ are even. It is known that these matrices satisfy the following relations:[3]

$$M_0 + M_+ + M_- = 1\,, \tag{2.11}$$

$$\boldsymbol{v}_0 + \boldsymbol{v}_+ + \boldsymbol{v}_- = 0\,, \tag{2.12}$$

$$[M_0, M_1] = 0\,, \tag{2.13}$$

$$M_1^2 = (1 - M_0)(1 + 3M_0)\,, \tag{2.14}$$

$$3(1 - M_0)\boldsymbol{v}_0 + M_1\boldsymbol{v}_1 = 0\,, \tag{2.15}$$

$$3M_1\boldsymbol{v}_0 + (1 + 3M_0)\boldsymbol{v}_1 = 0\,. \tag{2.16}$$

The Neumann coefficients $M_0, M_1, \boldsymbol{v}_0$ and $\boldsymbol{v}_1$ can be represented by a simpler ones K_1 and $\mathbf{u}$ [10, 13]:

$$M_0 = -\frac{1}{1 + 2\cosh(K_1\pi/2)}\,, \tag{2.17}$$

$$M_1 = \frac{2\sinh(K_1\pi/2)}{1 + 2\cosh(K_1\pi/2)}\,, \tag{2.18}$$

$$\boldsymbol{v}_0 = -\frac{1}{3}(1 + 3M_0)\mathbf{u}\,, \tag{2.19}$$

$$\boldsymbol{v}_1 = M_1\mathbf{u}\,, \tag{2.20}$$

where the matrix K_1 and the vector $\mathbf{u}$ are given by

$$[K_1]_{mn} = -\sqrt{n(n+1)}\,\delta_{m,n+1} - \sqrt{n(n-1)}\,\delta_{m,n-1}\,, \tag{2.21}$$

$$[\mathbf{u}]_n = \frac{1}{\sqrt{n}}\cos\left(\frac{n\pi}{2}\right) = \begin{cases} \dfrac{(-1)^{\frac{n}{2}}}{\sqrt{n}} & (n : \text{even}) \\ 0 & (n : \text{odd.}) \end{cases} \tag{2.22}$$

[3] Various formulas for the ghost Neumann coefficients are summarized in appendix A.

Here, K_1 is the matrix representation of the Virasoro algebra $K_1 = L_1 + L_{-1}$ and is twist-odd, $CK_1C = -K_1$. The eigenvalue problem of the matrix K_1 has been solved in [10]. The zero eigenvector $\mathbf{f}^{(0)}$ of K_1 satisfying $K_1\mathbf{f}^{(0)} = 0$ will be important in later discussions. It is a twist-odd vector and explicitly given by

$$[\mathbf{f}^{(0)}]_n = \frac{1}{\sqrt{n}}\sin\Bigl(\frac{n\pi}{2}\Bigr) = \begin{cases} 0 & (n : \text{even}) \\ \dfrac{(-1)^{\frac{n-1}{2}}}{\sqrt{n}} & (n : \text{odd.}) \end{cases} \tag{2.23}$$

A classical solution of VSFT which satisfies

$$\mathcal{Q}\Psi_{\mathrm{c}} + \Psi_{\mathrm{c}} * \Psi_{\mathrm{c}} = 0\,, \tag{2.24}$$

and is expected to represent a D25-brane has been found [5, 2, 6]:

$$|\Psi_{\mathrm{c}}\rangle = \mathcal{N}_{\mathrm{c}} b_0 \exp\Bigl(-\frac{1}{2}\boldsymbol{a}^\dagger\cdot CT\boldsymbol{a}^\dagger - \boldsymbol{b}^\dagger\cdot C\widetilde{T}\boldsymbol{c}^\dagger\Bigr)|0\rangle\,, \tag{2.25}$$

where $\mathcal{N}_{\mathrm{c}}$ is a normalization factor and T is an infinite dimensional twist-even matrix. The matrix T is a solution to

$$T = T\star T\,, \tag{2.26}$$

where $T\star T$ is defined by

$$T\star T \equiv M_0 + (M_+, M_-)(1 - T\mathcal{M})^{-1}T\begin{pmatrix} M_- \\ M_+ \end{pmatrix}, \tag{2.27}$$

with

$$\mathcal{M} = \begin{pmatrix} M_0 & M_+ \\ M_- & M_0 \end{pmatrix}. \tag{2.28}$$

Eq. (2.26) can be solved by using the relations (2.11)–(2.16) and assuming the commutativity among T and $M_{0,\pm}$. We take the following one as a solution of (2.26):

$$T = \frac{1}{2M_0}\Bigl(1 + M_0 - \sqrt{(1-M_0)(1+3M_0)}\Bigr)\,. \tag{2.29}$$

The equation of motion (2.24) also fixes the coefficient $\boldsymbol{f}$ in $\mathcal{Q}$ (2.2) as follows [6, 14, 15]:

$$\boldsymbol{f} = (1-\widetilde{T})^{-1}\left[\widetilde{\boldsymbol{v}}_0 + (\widetilde{M}_+, \widetilde{M}_-)(1-\widetilde{T}\widetilde{\mathcal{M}})^{-1}\widetilde{T}\begin{pmatrix}\widetilde{\boldsymbol{v}}_+ \\ \widetilde{\boldsymbol{v}}_-\end{pmatrix}\right], \tag{2.30}$$

where $\widetilde{T}$ is given by (2.29) with M_0 replaced by $\widetilde{M_0}$ (see appendix A for details).

Now, we expand the string field Ψ around the classical solution Ψ_{c}:

$$\Psi = \Psi_{\mathrm{c}} + \Phi\,. \tag{2.31}$$

The linearized equation of motion for the fluctuation Φ is[4]

$$\mathcal{Q}_{\mathrm{B}}\Phi \equiv \mathcal{Q}\Phi + \Psi_{\mathrm{c}} * \Phi + \Phi * \Psi_{\mathrm{c}} = 0\,. \tag{2.32}$$

[4]The action of the BRST operator $\mathcal{Q}_{\mathrm{B}}$ around Ψ_{c} on a generic string field A is defined by $\mathcal{Q}_{\mathrm{B}}A \equiv \mathcal{Q}A + \Psi_{\mathrm{c}} * A - (-)^{|A|}A * \Psi_{\mathrm{c}}$ with $|A| = 0$ (1) if A is Grassmann-even (-odd).

We take the following state $|\Phi_{\rm t}\rangle$ as the tachyon fluctuation mode [6]:

$$|\Phi_{\rm t}\rangle = \frac{1}{\mathcal{N}_{\rm c}} \exp\Bigl(-\sum_{n\geq 1} t_n a_n^\dagger a_0 + ip\cdot x\Bigr)|\Psi_{\rm c}\rangle\,. \tag{2.33}$$

The linearized equation of motion (2.32) for the tachyon state (2.33) determines the condition for the infinite dimensional vector $\boldsymbol{t}$ as

$$\boldsymbol{t} = T\star\boldsymbol{t}\,, \tag{2.34}$$

with the vector $T\star\boldsymbol{t}$ defined by

$$T\star\boldsymbol{t} \equiv \boldsymbol{v}_0 - \boldsymbol{v}_+ + (M_+, M_-)(1-T\mathcal{M})^{-1}\left[T\begin{pmatrix}\boldsymbol{v}_+ - \boldsymbol{v}_- \\ \boldsymbol{v}_- - \boldsymbol{v}_0\end{pmatrix} + \begin{pmatrix}0\\ \boldsymbol{t}\end{pmatrix}\right]. \tag{2.35}$$

Using the relations (2.11)–(2.16) and the twist-even property of $\boldsymbol{t}$,

$$C\boldsymbol{t} = \boldsymbol{t}\,, \tag{2.36}$$

(2.34) is solved to give

$$\boldsymbol{t} = 3(1+T)(1+3M_0)^{-1}\boldsymbol{v}_0\,. \tag{2.37}$$

The linearized equation of motion (2.32) also determines the mass squared of the state (2.33):

$$(\text{tachyon mass})^2 = -1\,. \tag{2.38}$$

This is the correct mass squared for the open string tachyon (we are taking the convention of $\alpha' = 1$).

3. Singular behavior around the zero-mode of K_1

In the calculation of the tachyon mass and other quantities in the oscillator formalism of VSFT, we are often faced with the phenomenon where the zero-mode of K_1 plays an important role. This is the case also in our analysis of the gauge structure in VSFT. Here, we briefly review this phenomenon. See [9, 13] for more details.

Various quantities in VSFT are expressed in terms of Neumann coefficients. For example, the tachyon mass squared is given by $-\ln 2/G$ with

$$G = \frac{9\sqrt{3}}{32}\,\boldsymbol{v}_1^{\rm T}\left(\frac{1-M_0}{\sqrt{1+3M_0}} - M_1\frac{1}{(1+3M_0)^{3/2}}M_1\right)\boldsymbol{v}_1\,. \tag{3.1}$$

This quantity is exactly equal to zero if we naively use the non-linear relations (2.13)–(2.16) for the Neumann coefficients. However, the expression (3.1) is in fact *indefinite* because the matrix M_0 has the eigenvalue $-1/3$. The corresponding eigenvector is the zero-mode $\mathbf{f}^{(0)}$ of K_1.

In the correct treatment, we must regularize G properly, and we use the finite-size matrix regularization following [9, 13, 11]. In this regularization, we truncate the infinite dimensional matrices M_0, M_1 into $L\times L$ ones. After this regularization, the non-linear relations (2.13)–(2.16) no longer hold. Because only the parts around the singularity at $K_1 = 0$ make finite contributions to the quantity G, it is sufficient to Laurent-expand around $K_1 = 0$:

$$(1 + 3M_0)|_L \simeq \frac{\pi^2}{12} K_1^2|_L \,, \tag{3.2}$$

$$M_1|_L \simeq \frac{\pi}{3} K_1|_L \,, \tag{3.3}$$

$$\boldsymbol{v}_1|_L \simeq \frac{\pi}{3} (K_1 \mathbf{u})|_L \,. \tag{3.4}$$

Note that the truncation of the squared matrix, $K_1^2|_L$, is different from the square of the truncated matrix, $(K_1|_L)^2$. With these manipulations, G is expressed as follows:

$$G = \frac{\pi}{4} (\mathbf{u}^{\mathrm{T}} K_1)|_L \left(\frac{1}{\sqrt{K_1^2|_L}} - K_1|_L \left(\frac{1}{\sqrt{K_1^2|_L}} \right)^3 K_1|_L \right) (K_1 \mathbf{u})|_L \,. \tag{3.5}$$

Taking the limit $L \to \infty$ after calculation with the finite-size matrices, we get the finite and expected value $G = \ln 2$. The reason why the quantity G vanishes upon use of the non-linear relations of the Neumann coefficients is reduced to the degeneracy of the eigenvalues of M_0 between twist-odd sector and twist-even one [9]. Therefore, the phenomenon that a quantity such as G which naively vanishes due to this degeneracy acquires a non-zero value is called twist anomaly.

As we have seen in this section, the singular behavior around the zero-mode of K_1 makes VSFT around the classical solution non-trivial. Besides the above example, this zero-mode plays a key role in constructing the higher excitation modes around the classical solution [12, 7]. In the next section, we shall see that a similar phenomenon happens in the analysis of the gauge structure of VSFT.

4. Gauge structure of VSFT

If the classical solution (2.25) of VSFT describes a D25-brane, VSFT expanded around it has to reproduce the ordinary open string theory. In particular, it must have the gauge structure of string theory. This section is devoted to the analysis of the gauge structure of VSFT expanded around the classical solution and is the main part of this paper. Here, we concentrate on the gauge structure of the massless vector field mentioned in the introduction; the gauge transformation $\delta A_\mu = \partial_\mu \lambda$ and the transversality condition $\partial_\mu A^\mu = 0$.

First, let us study the gauge transformation on the massless vector field. The massless vector state around the classical solution in VSFT is constructed in [12] and is given by

$$|\Phi_{\mathrm{v}}\rangle = \zeta^\mu \bar{\mathbf{f}}^{(0)} \cdot \boldsymbol{a}_\mu^\dagger |\Phi_{\mathrm{t}}\rangle \,, \tag{4.1}$$

where ζ^μ is a polarization vector and $\bar{\mathbf{f}}^{(0)}$ is the normalized zero eigenvector of K_1. The VSFT gauge transformation (2.3) is rewritten for the fluctuation Φ around Ψ_{c} as

$$\delta_\Lambda \Phi = \mathcal{Q}_{\mathrm{B}} \Lambda + \Phi * \Lambda - \Lambda * \Phi \,. \tag{4.2}$$

Here, we are interested in the inhomogeneous part:

$$\mathcal{Q}_{\mathrm{B}}\Lambda = \mathcal{Q}\Lambda + \Psi_{\mathrm{c}} * \Lambda - \Lambda * \Psi_{\mathrm{c}}\,. \tag{4.3}$$

As a candidate Λ which induces the massless vector gauge transformation, we take

$$|\Lambda\rangle = \boldsymbol{h}\cdot\boldsymbol{b}^{\dagger}|\Phi_{\mathrm{t}}\rangle\,. \tag{4.4}$$

Since the massless vector state (4.1) is twist-odd, the gauge transformation (4.3) and hence Λ itself must be so. Therefore the vector $\boldsymbol{h}$ in (4.4) must be a twist-odd vector satisfying $C\boldsymbol{h} = -\boldsymbol{h}$.

The term $|\Psi_{\mathrm{c}} * \Lambda\rangle$ in (4.3) is explicitly given in terms of the Neumann coefficients as follows:

$$\begin{aligned}|\Psi_{\mathrm{c}} * \Lambda\rangle = &- 2^{-p^2}\Biggl\{(0,\boldsymbol{h})(1-\widetilde{\mathcal{M}}\widetilde{T})^{-1}\begin{pmatrix}\widetilde{\boldsymbol{v}}_{+}\\ \widetilde{\boldsymbol{v}}_{-}\end{pmatrix} + \\ &+\left[(0,\boldsymbol{h})(1-\widetilde{\mathcal{M}}\widetilde{T})^{-1}\begin{pmatrix}\widetilde{M}_{-}\\ \widetilde{M}_{+}\end{pmatrix}\boldsymbol{b}^{\dagger}\right]\left[c_0 + \boldsymbol{c}^{\dagger}(1-\widetilde{T})\boldsymbol{f}\right]\Biggr\}\times \\ &\times b_0 \exp\left\{-(T\star\boldsymbol{t})\cdot\boldsymbol{a}^{\dagger}a_0 + ip\cdot x - \frac{1}{2}\boldsymbol{a}^{\dagger}\cdot C(T\star T)\boldsymbol{a}^{\dagger} - \boldsymbol{b}^{\dagger}\cdot C(\widetilde{T}\,\widetilde{\star}\,\widetilde{T})\boldsymbol{c}^{\dagger}\right\}|0\rangle\,,\end{aligned} \tag{4.5}$$

where $T\star\boldsymbol{t}$ and $T\star T$ are defined by (2.35) and (2.27), respectively. The other term $-|\Lambda*\Psi_{\mathrm{c}}\rangle$ in (4.3) is given by (4.5) with $(0,\boldsymbol{h})$ and $T\star\boldsymbol{t}$ replaced by $(\boldsymbol{h},0)$ and $C(T\star\boldsymbol{t})$, respectively. Namely, $|\Lambda * \Psi_{\mathrm{c}}\rangle$ is the twist transform of $|\Psi_{\mathrm{c}} * \Lambda\rangle$.[5] Note that the state in the last line of (4.5) is the tachyon state (2.33) with the replacements $(\boldsymbol{t}, T, \widetilde{T}) \to (T\star\boldsymbol{t}, T\star T, \widetilde{T}\,\widetilde{\star}\,\widetilde{T})$.

In deriving (4.5), we have not used any one of the non-linear relations (2.11)–(2.16) which are potentially invalid in the finite-size regularization, except at 2^{-p^2} which has been obtained by applying the calculation of section 3 to its original expression e^{-Gp^2}. Although we need to carefully treat the Neumann coefficients as described in the previous section, let us first simplify (4.5) by naive manipulations. Using the non-linear relations (2.11)–(2.16) and the equations (2.26) and (2.34), we have

$$\mathcal{Q}_{\mathrm{B}}|\Lambda\rangle|_{\mathrm{naive}} = (1-2^{-p^2})\boldsymbol{h}\cdot\boldsymbol{b}^{\dagger}\left(c_0 + [\boldsymbol{c}^{\dagger}(1-\widetilde{T})\boldsymbol{f}]\right)|\Phi_{\mathrm{t}}\rangle\,. \tag{4.6}$$

The part of (4.6) multiplied by 1 of $(1-2^{-p^2})$ has come from $\mathcal{Q}|\Lambda\rangle$. Since the state (4.6) does not contain the massless vector state (4.1), we cannot obtain the gauge transformation of the massless vector field in this naive treatment.

However, a careful treatment by taking into account the singularity at $K_1 = 0$ mentioned in section 3 will show that $\mathcal{Q}_{\mathrm{B}}\Lambda$ (4.3) does contain an additional term proportional to the massless vector state. To see this, let us first consider the following term in (4.5):

$$D = (0,\boldsymbol{h})(1-\widetilde{\mathcal{M}}\widetilde{T})^{-1}\begin{pmatrix}\widetilde{\boldsymbol{v}}_{+}\\ \widetilde{\boldsymbol{v}}_{-}\end{pmatrix} = -\frac{1}{2}\boldsymbol{h}\cdot(1,-1)(1-\widetilde{\mathcal{M}}\widetilde{T})^{-1}\begin{pmatrix}\widetilde{\boldsymbol{v}}_{+}\\ \widetilde{\boldsymbol{v}}_{-}\end{pmatrix}\,, \tag{4.7}$$

[5] The twist transformation property of the $*$-product of generic string fields A and B in our convention is $(\Omega A)*(\Omega B) = (-1)^{|A||B|}\Omega(B*A)$, where Ω is the twist transformation operator acting on string fields.

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where we have used the fact that $\boldsymbol{h}$ is twist-odd in obtaining the last expression. The corresponding term $(\boldsymbol{h},0)(1-\widetilde{\mathcal{M}}\widetilde{\mathcal{T}})^{-1}\begin{pmatrix}\widetilde{\boldsymbol{v}}_+\\ \widetilde{\boldsymbol{v}}_-\end{pmatrix}$ in $|\Lambda * \Psi_{\mathrm{c}}\rangle$ is equal to $-D$. In the naive expression (4.6), these two D terms have cancelled each other out since they are multiplied by the twist-even state $|\Phi_{\mathrm{t}}\rangle$. Using the non-linear relations for the Neumann coefficients, D is calculated to give

$$D=-\frac{1}{2}\boldsymbol{h}\cdot[(1-\widetilde{M}_0)(1+\widetilde{T})]^{-1}(1-\widetilde{T})\widetilde{\boldsymbol{v}}_1\,. \tag{4.8}$$

The ghost Neumann matrix $\widetilde{M}_0$ has an eigenvector with eigenvalue 1, which corresponds to the zero-mode of K_1 (see appendix A). Therefore, this factor D is in general ill-defined and needs to be regularized. As is shown in appendix B, D is of the order of $\sqrt{\ln L}$ in the finite-size regularization,[6]

$$D\simeq-\frac{1}{\pi}\sqrt{\ln L}\,, \tag{4.9}$$

if we choose as the vector $\boldsymbol{h}$ the following one,

$$\boldsymbol{h}=E^{-1}\bar{\mathbf{f}}^{(0)}\,, \tag{4.10}$$

with

$$E_{mn}=\delta_{mn}\sqrt{n}\,. \tag{4.11}$$

For a generic twist-odd vector $\boldsymbol{h}$ which does not contain the zero-mode $\mathbf{f}^{(0)}$, the factor D is less divergent than $\sqrt{\ln L}$.

Recall that in deriving the naive expression (4.6), the factor D has been canceled out between $|\Psi_{\mathrm{c}} * \Lambda\rangle$ and $|\Lambda * \Psi_{\mathrm{c}}\rangle$. However, since D is divergent for $\boldsymbol{h}$ of (4.10), it can couple with some infinitesimal twist-odd quantity to produce finite effects which were overlooked in deriving (4.6). In (4.6) we have replaced $T\star\boldsymbol{t}$ in the exponent of (4.5) with $\boldsymbol{t}$ by using $\boldsymbol{t}=T\star\boldsymbol{t}$ (2.34). However, as was pointed out in [11], (2.34) is not exactly satisfied by the solution (2.37) in the finite-size regularization. In fact, the infinitesimal twist-odd part exists in $T\star\boldsymbol{t}$ though $\boldsymbol{t}$ is twist-even. This infinitesimal twist-odd part multiplied by D can produce a finite contribution to $\mathcal{Q}_B\Lambda$. Besides the twist-odd part of $T\star\boldsymbol{t}$, there are other infinitesimal differences between the exponent of (4.5) and that of $|\Phi_{\mathrm{t}}\rangle$. For example, there is an infinitesimal difference between $T\star T$ and T in this regularization. But these differences are all twist-even and do not contribute to $\mathcal{Q}_{\mathrm{B}}\Lambda$.

Regularizing the Neumann coefficients to $L\times L$ ones and expanding them in K_1, the twist-odd part of $T\star\boldsymbol{t}$ is given by[7]

$$\begin{aligned}\frac{1-C}{2}T\star\boldsymbol{t}\Big|_L&\simeq-\frac{3}{4}\left(1-K_1|_L\frac{1}{K_1^2|_L}K_1|_L\right)\frac{\pi}{3}(K_1\mathbf{u})|_L\\ &=-\frac{\pi}{4}\left(\bar{\mathbf{f}}^{(0)}\big|_L\otimes\bar{\mathbf{f}}^{(0)}\big|_L\right)(K_1\mathbf{u})|_L\end{aligned}$$

[6]As seen from (B.14), the finite factor in D multiplying $\sqrt{\ln L}$ depends on the choice of $\bar{\boldsymbol{p}}_{n-\frac{1}{2}}$ as the finite-size version of $\bar{\mathbf{f}}^{(0)}$. In this section, we adopt $n=1$.

[7]The first expression of (4.12) is due to (3.13) of [11], $\Delta T\simeq(\pi/2)\sqrt{K_1^2}$, $\boldsymbol{t}\simeq-(3/2)(1/\sqrt{K_1^2})K_1\boldsymbol{v}_1$ and (3.4).

$$= -\frac{\pi}{4}\sqrt{\frac{2}{\ln L}}\,\bar{\mathbf{f}}^{(0)}\big|_L\,, \tag{4.12}$$

where we have used the equality (C.14) and the fact that $(K_1\mathbf{u})\big|_L = (1,0,0,\dots,0)$. One might think that it is not clear whether the leading contribution of the twist-odd part of $T\star\boldsymbol{t}$ is given by the expansion in K_1. However, this is justified by considering the inner product of $[(1-C)/2]\,T\star\boldsymbol{t}|_L$ with a suitable basis of the vector space, and carrying out the same argument as in section 3 for G. The infinitesimal factor in (4.12) and the divergent one D appear in $|\Psi_{\rm c}*\Lambda\rangle - |\Lambda*\Psi_{\rm c}\rangle$ in the following way to give a finite contribution:

$$De^{\varepsilon} - De^{-\varepsilon} \simeq 2D\varepsilon\,, \tag{4.13}$$

with

$$\varepsilon = -\frac{1-C}{2}(T\star\boldsymbol{t})\cdot\boldsymbol{a}^\dagger a_0\,. \tag{4.14}$$

Note that the $\sqrt{\ln L}$ factors are canceled in $D\varepsilon$ on the r.h.s. of (4.13). Including this new term, (4.3) is now given by

$$\mathcal{Q}_{\rm B}|\Lambda\rangle = (1-2^{-p^2})(\bar{\mathbf{f}}^{(0)}\cdot E^{-1}\boldsymbol{b}^\dagger)\left(c_0+[\boldsymbol{c}^\dagger(1-\widetilde{T})\boldsymbol{f}]\right)|\Phi_{\rm t}\rangle + 2^{-p^2}p^\mu\,\bar{\mathbf{f}}^{(0)}\cdot\boldsymbol{a}^\dagger_\mu|\Phi_{\rm t}\rangle\,. \tag{4.15}$$

The last term of (4.15), which is the massless vector state with the polarization proportional to the center-of-mass momentum p^μ, does generate the desired gauge transformation on the massless vector field. Recall that this term has emerged from $0\times\infty$ and hence is a kind of anomaly.

Next let us consider whether the transversality condition can arise by a similar analysis. Corresponding to (4.5) for $\mathcal{Q}_{\rm B}|\Lambda\rangle$, the term $|\Psi_{\rm c}*\Phi_{\rm v}\rangle$ in $\mathcal{Q}_{\rm B}|\Phi_{\rm v}\rangle$ for the vector state $|\Phi_{\rm v}\rangle = \boldsymbol{\zeta}^\mu\cdot\boldsymbol{a}^\dagger_\mu|\Phi_{\rm t}\rangle$ is given by

$$\begin{aligned}
|\Psi_{\rm c}*\Phi_{\rm v}\rangle = &-2^{-p^2}\Bigg\{\boldsymbol{a}^\dagger_\mu\cdot(M_+,M_-)\,(1-T\mathcal{M})^{-1}\begin{pmatrix}\boldsymbol{\zeta}^\mu\\ 0\end{pmatrix}+\\
&\qquad+\Big[(\boldsymbol{t},0)\mathcal{M}+(\boldsymbol{v}_- -\boldsymbol{v}_0,\boldsymbol{v}_+ -\boldsymbol{v}_-)\Big]\,(1-T\mathcal{M})^{-1}\begin{pmatrix}\boldsymbol{\zeta}^\mu\\ 0\end{pmatrix}a_0^\mu\Bigg\}\times\\
&\times\left(c_0+\boldsymbol{c}^\dagger(1-\widetilde{T})\boldsymbol{f}\right)\times\\
&\times b_0\exp\left\{-(T\star\boldsymbol{t})\cdot\boldsymbol{a}^\dagger a_0 + ip\cdot x - \frac{1}{2}\boldsymbol{a}^\dagger\cdot C(T\star T)\boldsymbol{a}^\dagger - \boldsymbol{b}^\dagger\cdot C(\widetilde{T}\,\widetilde{\star}\,\widetilde{T})\boldsymbol{c}^\dagger\right\}|0\rangle\,.
\end{aligned} \tag{4.16}$$

For the genuine massless vector state with polarization ζ^μ, (4.1), we have $\boldsymbol{\zeta}^\mu = \zeta^\mu\bar{\mathbf{f}}^{(0)}$. Here for the moment, we treat $\boldsymbol{\zeta}^\mu$ as a generic twist-odd vector. First, the naive calculation gives

$$\mathcal{Q}_{\rm B}|\Phi_{\rm v}\rangle|_{\rm naive} = (1-2^{-p^2})\left\{\boldsymbol{\zeta}^\mu\cdot\boldsymbol{a}^\dagger_\mu\left(c_0+[\boldsymbol{c}^\dagger(1-\widetilde{T})\boldsymbol{f}]\right)\right\}|\Phi_{\rm t}\rangle\,, \tag{4.17}$$

implying that the linearized equation of motion of $\Phi_{\rm v}$ is satisfied by $p^2=0$ for an arbitrary $\boldsymbol{\zeta}^\mu$ [6]. However, corresponding to D (4.8) for the gauge transformation, there exists a dangerous factor also in the present case. It is

$$\Big[(\boldsymbol{t},0)\mathcal{M}+(\boldsymbol{v}_- -\boldsymbol{v}_0,\boldsymbol{v}_+ -\boldsymbol{v}_-)\Big]\,(1-T\mathcal{M})^{-1}\begin{pmatrix}\boldsymbol{\zeta}^\mu\\ 0\end{pmatrix} = -\frac{1}{2}\boldsymbol{v}_1\cdot(1-T)\left[(1-M_0)(1+T)\right]^{-1}\boldsymbol{\zeta}^\mu\,, \tag{4.18}$$

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where the second expression has been obtained by using the non-linear relations for the Neumann coefficients and that $\boldsymbol{\zeta}^\mu$ is twist-odd. Since we have $T=-1$ at $K_1=0$, this quantity (4.18) with $\boldsymbol{\zeta}^\mu=\zeta^\mu\bar{\mathbf{f}}^{(0)}$ is $\sqrt{\ln L}$ divergent in the finite-size regularization in the same manner as for D. In fact, the singular part of (4.18) is the same as that of D. Then, taking into account this divergent nature of (4.18) and the infinitesimal twist-odd part of $T\star\boldsymbol{t}$, we find that $Q_{\rm B}|\Phi_{\rm v}\rangle$ with the anomaly contribution included is given by

$$\mathcal{Q}_{\rm B}|\Phi_{\rm v}\rangle=\zeta^\mu\left[(1-2^{-p^2})\delta^\nu_\mu+\sqrt{2}\,2^{-p^2}p_\mu p^\nu\right]\bar{\mathbf{f}}^{(0)}\cdot\boldsymbol{a}^\dagger_\nu\left(c_0+[\boldsymbol{c}^\dagger(1-\widetilde{T})\boldsymbol{f}]\right)|\Phi_{\rm t}\rangle\,. \tag{4.19}$$

Eq. (4.19) implies that $\mathcal{Q}_{\rm B}\Phi_{\rm v}=0$ is satisfied if only one equation for the polarization,

$$\left(\left(1-2^{-p^2}\right)\eta_{\mu\nu}+\sqrt{2}\,2^{-p^2}p_\mu p_\nu\right)\zeta^\nu=0\,, \tag{4.20}$$

is satisfied. The effect of the anomaly is merely to add the $p_\mu p_\nu$ term to the naive mass-shell condition $\left(1-2^{-p^2}\right)\zeta^\mu=0$, namely, to change the gauge of the equation. Therefore, even if we take into account the anomaly contribution, the transversality condition, $p_\mu\zeta^\mu=0$, cannot be obtained.

As an example of ζ^μ which satisfies (4.20) but is not subject to the transversality condition, we have the following longitudinal mode,[8]

$$\zeta^\mu_{\rm L}=\left[\frac{\bar{p}^\mu\delta(p^2)}{(p\cdot\bar{p})}+\beta p^\mu\delta'(p^2)\right]\theta(p^0)f(\boldsymbol{p})\,, \tag{4.21}$$

where $f(\boldsymbol{p})$ is an arbitrary regular function and $\beta=(1+\ln 2/\sqrt{2})^{-1}$. Here, $\bar{p}^\mu$ is defined by $\bar{p}^\mu=(p,0,\dots,0,-p)$ in the frame where p^μ is given by $p^\mu=(p,0,\dots,0,p)$. This solution $\zeta^\mu_{\rm L}$ does not satisfy the transversality condition:

$$p_\mu\zeta^\mu_{\rm L}=(1-\beta)\delta(p^2)\theta(p^0)f(\boldsymbol{p})\,. \tag{4.22}$$

Since the longitudinal mode (4.21) contains the derivative of δ-function, one might think that it is not normalizable and hence we can ignore it in the analysis of the spectrum. However, it is normalizable as seen below. Let us consider the following inner product for general polarization vectors:

$$\langle\zeta,\zeta'\rangle\equiv\int d^{26}p\int d^{26}p'\,\zeta^\mu(p)\zeta'^\nu(p')\eta_{\mu\nu}\delta^{25}(\boldsymbol{p}-\boldsymbol{p}')\,. \tag{4.23}$$

Obviously, the norm defined by this inner product is convergent and gives the usual one for the transverse modes, $\zeta^\mu\propto\delta(p^2)$. For the longitudinal mode $\zeta^\mu_{\rm L}$ (4.21), its norm is given by

$$\langle\zeta_{\rm L},\zeta_{\rm L}\rangle=\frac{\beta^2+2\beta}{16}\int d^{25}\boldsymbol{p}\,\frac{|f(\boldsymbol{p})|^2}{|\boldsymbol{p}|^4}\,. \tag{4.24}$$

In obtaining (4.24), we have repeatedly carried out integrations by parts for terms containing $\delta'(p^2)$. If we choose a well-defined function, e.g. a gaussian, as $f(\boldsymbol{p})$, this norm is finite. Thus, the longitudinal mode (4.21) is a normalizable solution of (4.20) and must be included in the spectrum. We need the transversality condition to exclude this spurious mode.

[8]We would like to thank T. Kugo for discussions on this point.

5. Summary

In this paper, we have investigated the gauge structure of VSFT expanded around the classical solution which is expected to represent a D25-brane. We obtained the correct gauge transformation for the massless vector state as a kind of anomaly. The infinite factor from the ghost part and the infinitesimal one of the matter part cooperate to make a finite contribution to the gauge transformation. Unfortunately, the transversality condition cannot be gained even if we take into account the anomaly.

The failure in obtaining the transversality condition is a serious problem. However, it could give a hint to the resolution of other problems in the oscillator formulation of VSFT, in particular, the problem of obtaining the correct D-brane tension [9, 16, 13, 11]. The discussion in this paper is limited to the massless vector state. It is also an interesting problem to extend our analysis to the higher level fluctuation states [7].

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A. Ghost Neumann coefficients

We summarize the properties of the ghost Neumann coefficients which are used in the calculations in this paper. The ghost Neumann coefficients satisfy the following relations which correspond to (2.11)–(2.16) for the matter ones:

$$\widetilde{M}_0 + \widetilde{M}_+ + \widetilde{M}_- = 1\,, \tag{A.1}$$

$$\widetilde{\boldsymbol{v}}_0 + \widetilde{\boldsymbol{v}}_+ + \widetilde{\boldsymbol{v}}_- = 0\,, \tag{A.2}$$

$$[\widetilde{M}_0, \widetilde{M}_1] = 0\,, \tag{A.3}$$

$$\widetilde{M}_1^2 = (1-\widetilde{M}_0)(1+3\widetilde{M}_0)\,, \tag{A.4}$$

$$(1+3\widetilde{M}_0)\widetilde{\boldsymbol{v}}_0 - \widetilde{M}_1\widetilde{\boldsymbol{v}}_1 = 0\,, \tag{A.5}$$

$$\widetilde{M}_1\widetilde{\boldsymbol{v}}_0 - (1-\widetilde{M}_0)\widetilde{\boldsymbol{v}}_1 = 0\,. \tag{A.6}$$

The matrix $\widetilde{T}$ in (2.25) is a solution to $\widetilde{T} = \widetilde{T}\,\widetilde{\star}\,\widetilde{T}$ with $\widetilde{\star}$ defined by (2.27) with all the matrices replaced by tilded ones. It is given by

$$\widetilde{T} = \frac{1}{2\widetilde{M}_0}\left(1+\widetilde{M}_0 - \sqrt{(1-\widetilde{M}_0)(1+3\widetilde{M}_0)}\right). \tag{A.7}$$

The eigenvalue distribution of $\widetilde{M}_0$ is in the range $[0,1]$, and $\widetilde{M}_0 = 0$ (1) corresponds to $\widetilde{T} = 0$ (1).

The ghost Neumann coefficients are related to the matter ones by [17]

$$\widetilde{M}_0 = -E\frac{M_0}{1+2M_0}E^{-1}\,, \tag{A.8}$$

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$$\widetilde{M}_1 = E\frac{M_1}{1+2M_0}E^{-1}\,, \tag{A.9}$$

where the matrix E is defined by (4.11). Using these relations, the ghost Neumann coefficients can be expressed in terms of E, K_1 (2.21) and $\mathbf{u}$ (2.22):

$$\widetilde{M}_0 = E\frac{1}{2\cosh(K_1\pi/2)-1}E^{-1}\,, \tag{A.10}$$

$$\widetilde{M}_1 = E\frac{2\sinh(K_1\pi/2)}{2\cosh(K_1\pi/2)-1}E^{-1}\,, \tag{A.11}$$

$$\widetilde{\boldsymbol{v}}_0 = (1-\widetilde{M}_0)E\mathbf{u}\,, \tag{A.12}$$

$$\widetilde{\boldsymbol{v}}_1 = \widetilde{M}_1 E\mathbf{u}\,. \tag{A.13}$$

Note that $K_1 = 0$ corresponds to $\widetilde{M}_0 = 1$. Laurent-expansion around $K_1 = 0$ in the finite-size regularization gives

$$(1-\widetilde{M}_0)|_L \simeq \frac{\pi^2}{4}EK_1^2|_L E^{-1}\,, \tag{A.14}$$

$$\widetilde{M}_1|_L \simeq \pi E K_1|_L E^{-1}\,, \tag{A.15}$$

$$\widetilde{\boldsymbol{v}}_1|_L \simeq \pi E(K_1\mathbf{u})|_L\,, \tag{A.16}$$

$$1-\widetilde{T}(\widetilde{M}_1|_L) \simeq \frac{\pi}{2}E\sqrt{K_1^2|_L}E^{-1}\,, \tag{A.17}$$

where $\widetilde{T}(\widetilde{M}_1|_L)$ in (A.17) is $\widetilde{T}$ (A.7) with $\widetilde{M}_0$ replaced by the cutoff one $\widetilde{M}_0|_L$.

B. Calculation of the divergent factor D

In this appendix, we calculate the divergent factor D (4.8). Using the formulas in appendix A, D is expanded around $K_1 = 0$ as

$$D \simeq -\frac{1}{2}\boldsymbol{h}\cdot E\frac{1}{\sqrt{K_1^2\big|_L}}(K_1\mathbf{u})\big|_L\,, \tag{B.1}$$

where we have kept only the most singular term. As shown in [13], $1/\sqrt{K_1^2\big|_L}$ has the following representation:

$$\frac{1}{\sqrt{K_1^2\big|_L}} = \begin{pmatrix} P_{\mathrm{H}}\Lambda_{\mathrm{H}}^{-1}P_{\mathrm{H}}^{\mathrm{T}} & 0 \\ 0 & Q_{\mathrm{I}}\Lambda_{\mathrm{I}}^{-1}Q_{\mathrm{I}}^{\mathrm{T}} \end{pmatrix}. \tag{B.2}$$

Here, we have changed the rows and columns of the matrix and collected the elements with odd-odd indices in the upper-left block, and vice versa. The matrix P_{H} (Q_{I}) is constructed from the normalized twist-odd (even) eigenvectors $\bar{\boldsymbol{p}}_{n-\frac{1}{2}}$ ($\bar{\boldsymbol{q}}_n$) of $K_1^2\big|_L$ and Λ_{H} (Λ_{I}) is the diagonal matrices consisting of the corresponding eigenvalues $\kappa_{n-\frac{1}{2}}$ (κ_n):

$$P_{\mathrm{H}} \equiv (\bar{\boldsymbol{p}}_{\frac{1}{2}}, \bar{\boldsymbol{p}}_{\frac{3}{2}}, \ldots)\,, \tag{B.3}$$

$$Q_{\mathrm{I}} \equiv (\bar{\boldsymbol{q}}_1, \bar{\boldsymbol{q}}_2, \dots)\,, \tag{B.4}$$

$$\Lambda_{\mathrm{H}} \equiv \mathrm{diag}(\kappa_{n-\frac{1}{2}})\,, \tag{B.5}$$

$$\Lambda_{\mathrm{I}} \equiv \mathrm{diag}(\kappa_n)\,, \tag{B.6}$$

with

$$K_1^2 \bar{\boldsymbol{p}}_{n-\frac{1}{2}} = \kappa_{n-\frac{1}{2}}^2 \bar{\boldsymbol{p}}_{n-\frac{1}{2}}\,, \tag{B.7}$$

$$K_1^2 \bar{\boldsymbol{q}}_n = \kappa_n^2 \bar{\boldsymbol{q}}_n\,, \tag{B.8}$$

$$C \bar{\boldsymbol{p}}_{n-\frac{1}{2}} = -\bar{\boldsymbol{p}}_{n-\frac{1}{2}}\,, \tag{B.9}$$

$$C \bar{\boldsymbol{q}}_n = \bar{\boldsymbol{q}}_n\,. \tag{B.10}$$

The bar on the vectors represents that they are normalized.

Before evaluating the divergent factor D, we must fix the twist-odd vector $\boldsymbol{h}$. We adopt as $E\boldsymbol{h}$ any one of the twist-odd eigenvectors $\bar{\boldsymbol{p}}_{n-\frac{1}{2}}$, which constitute an orthonormal basis of the twist-odd subspace in the L dimensional vector space. Using the fact that $(K_1 \mathbf{u})\big|_L = (1, 0, 0, \dots, 0)$ and that the first component of $\bar{\boldsymbol{p}}_{n-\frac{1}{2}}$ is given by [13]

$$[\bar{\boldsymbol{p}}_{n-\frac{1}{2}}]_1 = \sqrt{\frac{2\pi\kappa_{n-\frac{1}{2}}}{\sinh\bigl(\kappa_{n-\frac{1}{2}}\pi/2\bigr)\ln L}}\,. \tag{B.11}$$

D is evaluated as

$$\begin{aligned} D &= -\frac{1}{2}\sum_m (\bar{\boldsymbol{p}}_{n-\frac{1}{2}} \cdot \bar{\boldsymbol{p}}_{m-\frac{1}{2}}) \frac{1}{\kappa_{m-\frac{1}{2}}} [\bar{\boldsymbol{p}}_{m-\frac{1}{2}}]_1 \\ &= -\frac{1}{2}\frac{1}{\kappa_{n-\frac{1}{2}}} [\bar{\boldsymbol{p}}_{n-\frac{1}{2}}]_1 \\ &= -\frac{1}{2}\sqrt{\frac{2\pi}{\kappa_{n-\frac{1}{2}} \sinh\bigl(\kappa_{n-\frac{1}{2}}\pi/2\bigr)\ln L}}\,. \end{aligned} \tag{B.12}$$

For $n \ll \ln L$, the eigenvalue $\kappa_{n-\frac{1}{2}}$ is given by

$$\kappa_{n-\frac{1}{2}} = \frac{2\pi}{\ln L}\left(n - \frac{1}{2}\right), \tag{B.13}$$

and for such n, we have

$$D = -\frac{\sqrt{\ln L}}{2\pi(n - \frac{1}{2})}\,. \tag{B.14}$$

On the other hand, when n is of the same order or larger than $\ln L$, D is much smaller than $\sqrt{\ln L}$ and is insufficient to compensate the infinitesimal factor $\varepsilon \sim 1/\sqrt{\ln L}$. Because the eigenvectors $\bar{\boldsymbol{p}}_{n-\frac{1}{2}}$ for $n \ll \ln L$ approach the zero eigenvectors $\bar{\mathbf{f}}^{(0)}$ in the limit $L \to \infty$, we conclude that D generates the gauge transformation only if $E\boldsymbol{h}$ is proportional to $\mathbf{f}^{(0)}$.

C. The inverse of $K_1^2\big|_L$

In the calculation in (4.12), we must evaluate the quantity $K_1\big|_L(K_1^2\big|_L)^{-1}K_1\big|_L$. For this purpose, we present the explicit expression of the inverse of $K_1^2\big|_L$. Here, we consider the case of $L=$ odd. First, recall that the L-dimensional cutoff of the vectors $\mathbf{f}^{(0)}$ (2.23) and $\mathbf{u}$ (2.22) is given by

$$\mathbf{f}^{(0)}\big|_L = \left(1, 0, -\frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{5}}, 0, \ldots, 0, \frac{(-1)^{\frac{L-1}{2}}}{\sqrt{L}}\right)^{\mathrm{T}}, \tag{C.1}$$

$$\mathbf{u}\big|_L = \left(0, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{4}}, 0, \ldots, 0, \frac{(-1)^{\frac{L-1}{2}}}{\sqrt{L-1}}, 0\right)^{\mathrm{T}}. \tag{C.2}$$

In addition to these vectors, we define $\mathbf{f}_n$ and $\mathbf{u}_n$ which are the "truncated" vectors of $\mathbf{f}^{(0)}\big|_L$ and $\mathbf{u}\big|_L$:

$$\mathbf{f}_n = \left(1, 0, -\frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{5}}, 0, \ldots, 0, \frac{(-1)^{\frac{n-3}{2}}}{\sqrt{n-2}}, 0, \frac{(-1)^{\frac{n-1}{2}}}{\sqrt{n}}, 0, 0, \ldots, 0, 0\right)^{\mathrm{T}}, \tag{C.3}$$

for odd n and

$$\mathbf{u}_n = \left(0, 0, \ldots, 0, 0, \frac{(-1)^{\frac{n}{2}}}{\sqrt{n}}, 0, \frac{(-1)^{\frac{n+2}{2}}}{\sqrt{n+2}}, 0, \ldots, 0, \frac{(-1)^{\frac{L-1}{2}}}{\sqrt{L-1}}, 0\right)^{\mathrm{T}}, \tag{C.4}$$

for even n. Using the explicit representation of $K_1\big|_L$ and $K_1^2\big|_L$,

$$K_1\big|_L = \begin{pmatrix} 0 & -\sqrt{1\cdot 2} & & & & & \\ -\sqrt{1\cdot 2} & 0 & -\sqrt{2\cdot 3} & & & & \\ & -\sqrt{2\cdot 3} & 0 & -\sqrt{3\cdot 4} & & & \\ & & -\sqrt{3\cdot 4} & 0 & & \ddots & \\ & & & \ddots & & & \\ & & & & -\sqrt{(L-2)(L-1)} & 0 & -\sqrt{(L-1)L} \\ & & & & & -\sqrt{(L-1)L} & 0 \end{pmatrix}, \tag{C.5}$$

$$K_1^2\big|_L = \begin{pmatrix} 2\cdot 1^2 & 0 & 2\sqrt{1\cdot 3} & & & & \\ 0 & 2\cdot 2^2 & 0 & 3\sqrt{2\cdot 4} & & & \\ 2\sqrt{1\cdot 3} & 0 & 2\cdot 3^2 & 0 & \ddots & & \\ & 3\sqrt{2\cdot 4} & 0 & \ddots & & & \\ & & \ddots & & & \ddots & \\ & & & & 2(L-2)^2 & 0 & (L-1)\sqrt{L(L-2)} \\ & & & \ddots & 0 & 2(L-1)^2 & 0 \\ & & & & (L-1)\sqrt{L(L-2)} & 0 & 2L^2 \end{pmatrix}. \tag{C.6}$$

We can calculate the action of $K_1\big|_L$ and $K_1^2\big|_L$ on the truncated vectors:

$$K_1\big|_L\mathbf{f}_n = -(-1)^{\frac{n-1}{2}}\sqrt{n+1}\,\mathbf{e}_{n+1}, \tag{C.7}$$

$$K_1\big|_L\mathbf{u}_n = -(-1)^{\frac{n}{2}}\sqrt{n-1}\,\mathbf{e}_{n-1} - (-1)^{\frac{L-1}{2}}\sqrt{L}\,\mathbf{e}_L, \tag{C.8}$$

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$$K_1^2\big|_L \mathbf{f}_n = (-1)^{\frac{n-1}{2}}(n+1)\left(\sqrt{n}\mathbf{e}_n + \sqrt{n+2}\mathbf{e}_{n+2}\right), \tag{C.9}$$

$$K_1^2\big|_L \mathbf{u}_n = (-1)^{\frac{n}{2}}(n-1)\left(\sqrt{n}\mathbf{e}_n + \sqrt{n-2}\mathbf{e}_{n-2}\right) + (-1)^{\frac{L-1}{2}} L\sqrt{L-1}\mathbf{e}_{L-1}, \tag{C.10}$$

where $\mathbf{e}_n$ is the unit vector along the n-th direction; $[\mathbf{e}_n]_m = \delta_{nm}$. In (C.7)–(C.10), we have $\mathbf{e}_n = 0$ when $n > L$ and $n < 1$. Using these equations, we can show that the inverse of $K_1^2|_L$ is expressed as follows:

$$(K_1^2\big|_L)^{-1} = (\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3, \ldots, \mathbf{d}_{L-1}, \mathbf{d}_L), \tag{C.11}$$

where the vector $\mathbf{d}_n$ is defined by

$$\mathbf{d}_{2k+1} = \frac{(-1)^k}{\sqrt{2k+1}} \sum_{n=k+1}^{(L+1)/2} \frac{1}{2n}\mathbf{f}_{2n-1}, \tag{C.12}$$

$$\mathbf{d}_{2k} = \frac{(-1)^k}{\sqrt{2k}} \left(\sum_{n=1}^{k} \frac{\mathbf{u}_{2n}}{2n-1} - \frac{\sum_{m=1}^{k} \frac{1}{2m-1}}{\sum_{m=1}^{(L+1)/2} \frac{1}{2m-1}} \sum_{n=1}^{(L-1)/2} \frac{1}{2n-1}\mathbf{u}_{2n} \right). \tag{C.13}$$

From (C.7)–(C.10), (C.11), we finally obtain

$$\begin{aligned} \left[\mathbf{1}_L - K_1\big|_L \frac{1}{K_1^2\big|_L} K_1\big|_L \right]_{n,m} &= \left(\sum_{k=1}^{[L/2]+1} \frac{1}{2k-1} \right)^{-1} [\mathbf{f}^{(0)}\big|_L]_n [\mathbf{f}^{(0)}\big|_L]_m \\ &= [\bar{\mathbf{f}}^{(0)}\big|_L]_n [\bar{\mathbf{f}}^{(0)}\big|_L]_m . \end{aligned} \tag{C.14}$$

This is the projection operator which singles out the zero-mode of K_1. Eq. (C.14) holds also in the case $L = $ even.

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